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Universal Encoding for Unimodal Maps

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The standard encoding procedure to describe the chaotic orbits of unimodal maps is accurately investigated. We show that the grammatical rules of the underlying language can be easily classified in a compact form by means of a universal parameter τ. Two procedures to construct finite graphs which approximate non-Markovian cases are discussed, showing also the intimate relation with the corresponding construction of approximate Markov partitions. The convergence of the partial estimates of the topological entropy is discussed, proving that the error decreases exponentially with the length of the sequences considered. The rate is shown to coincide with the topological entropy h itself. Finally, a superconvergent method to estimate h is introduced.

KEY WORDS: Kneading theory; graphs; topological entropy; Markov partitions.

1. INTRODUCTION

Much progress has been recently made in the understanding of deterministic chaos. It is now widely accepted that a complete characterization of chaotic behavior is achieved (at least from a topological point of view) by first determining a generating partition which allows one to encode each trajectory in terms of a suitable symbol sequence, and then by finding the grammatical rules of the associated language.[1-3] This program can be considered as entirely accomplished whenever a possibly minimal graph,[4] generating all grammatically correct symbol sequences, is constructed. The difficulty of the task is directly related to the infinite grammar associated with typical chaotic systems. As a consequence, most authors have restricted their analysis to cases described by regular languages (i.e., loosely speaking, by a finite set of rules). A first relevant distinction must be made

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between one- and higher-dimensional maps. While a general scheme to build a directed graph has been already introduced for 1D maps,[1] only a preliminary approach has been developed for 2D maps, in terms of a suitable "pruning front."[2] More recently, a new technique based on progressive 1D approximations of a 2D map has been proposed,[5] which seems very promising to determine the grammatical rules.

Here, restricting our analysis to 1D unimodal maps,[6] we briefly review the technique introduced in ref. 1 and we describe a new technique to approximate the dynamics of a generic chaotic state in terms of "increasingly complicated" regular languages. The procedures resemble the approach to irrational numbers through truncated numbers of increasing length, and through rational numbers, respectively. The first method, less accurate, can, however, be easily extended to any dynamical system characterized by a finite generating partition. Both techniques owe their successful implementation to the existence of a universal parameter τ which summarizes in a compact form the precise ordering of chaotic states.[6] In particular, τ, which can be easily determined from the kneading sequence, allows one to reconstruct a general bifurcation diagram, which contains all the known information on the ordering of periodic cycles.[7] Periodic windows where h remains constant are easily identified as well. Our approach proceeds in the spirit of ref. 8, where a similar technique is introduced for iterated maps of the interval.

The existence of a universal parameter makes it possible to introduce a superconvergent method to compute the topological entropy h and, in turn, to quantify its local dependence on τ itself. The investigation of the resulting fractal structure of $h(\tau)$ allows one to quantify the error on the estimate of h, determined by using the grammatical rules extracted from the knowledge of all symbol sequences up to length n. The accuracy turns out to be of the order of $\exp(-hn)$ in generic cases.

Finally, the relation existing between the nodes of a suitably constructed graph (with the associated adjacency matrix) and the elements of an approximate Markov partition (with the associated transition matrix) are shown, thus solving a problem raised by Grassberger.[1]

The plan of the paper is a follows. In Section 2 we present the encoding procedure, discussing the admissible values of the universal parameter τ. In Section 3, the two different approaches to the construction of graphs are introduced, showing also the relations between truncated graphs and approximations with finite Markov partitions. In Section 4, we present a superconvergent method to determine the topological entropy h and discuss the fractal dependence of h on τ. In Section 5 the spectral properties of the graphs are briefly presented, showing in particular that the two procedures introduced in Section 3 yield the same topological spectrum. The relations between the characteristic polynomial of a graph and the standard statistical mechanical formalism applied to dynamical systems are also discussed. As a result, it is shown how the graph formalism is a powerful tool to investigate the analytic structure of topological ζ-functions. In Section 6 further remarks are given, mentioning also some open problems and future perspectives.

2. THE ENCODING PROCEDURE

To start with, let us recall some well-known facts. Without loss of generality, we can say that a continuous map f: $I \to I$ of the unit interval onto itself is *unimodal* if $f(0) = f(1) = 0$ and f has a unique maximum at c_0 with $0 < c_0 < 1$ (see Fig. 1). Such maps are perhaps the simplest models that simulate the complex behavior of real systems approaching chaos, at least insofar as they stretch and fold a given domain. For example, the logistic map $f_\mu = \mu x(1-x)$ belongs to this class, as well as other quadratic maps frequently studied in dynamical systems theory. Note that for a unimodal map f

$$\begin{aligned} f'(x) &\geqslant 0 \quad \text{if} \quad x < c_0 \\ f'(x) &\leqslant 0 \quad \text{if} \quad x > c_0 \end{aligned} \tag{2.1}$$

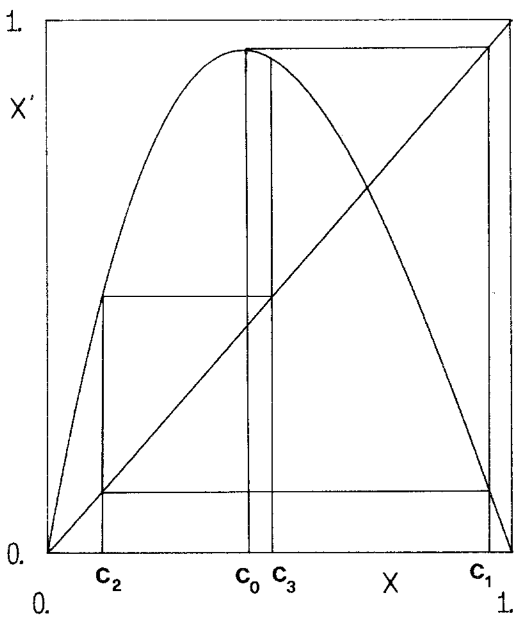


Fig. 1. Typical example of a unimodal map $x' = f(x)$, with the abscissa c_0 of the maximum and its first iterates c_i, explicitly reported.

In this case it is known that the orbit of the critical point c_0 determines, in some sense, the whole dynamics of the map. This observation is the starting point for the construction of that particular version of symbolic dynamics referred to as kneading theory (see ref. 6). In this context, if $x \in I$, we denote the *itinerary* of x under f through the sequence $S(x) = (s_0 s_1 s_2 \cdots)$, where s_i is either 0 or 1, depending upon whether $f^i(x)$ is $\leqslant c_0$ or $> c_0$, respectively (the critical point is taken to be 0 without loss of generality). The itinerary of c_1 $[=f(c_0)]$ is the *kneading sequence* K of f. Moreover, we say that a given sequence s of 0's and 1's is admissible (or "allowed") for f if there exists $x \in I$ such that $S(x) = s$. One of the simplest ways to decide whether a given sequence $s = (s_1, s_2, \ldots)$ is allowed or not is as follows.

First, let us denote the shifted sequence $(s_2, s_3, \ldots)$ by $\sigma(s)$, and the sequence $(t_1, t_2, \ldots)$ by $\tau(s)$, where the symbol t_k is the number (modulo 2) of 1's in s up to the position k, i.e.,

$$t_k = \sum_{i=1}^{k} s_i (\mathrm{mod}\ 2) \tag{2.2}$$

so that $t_k \in \{0, 1\}$ for each k.

Second, order the $\tau(s)$ as if they were binary representations of real numbers in $[0, 1]$; namely, represent them as

$$\tau(s) = 0.t_1 t_2 \cdots = \sum_{k=1}^{\infty} t^k 2^{-k}$$

It is straightforward to realize that if $\tau(s) = a$ (with $a \in [0, 1]$), then

$$\tau(\sigma(s)) = T(a) \tag{2.3}$$

where T is the tent map

$$T(x) = \begin{cases} 2x & \text{if} \quad x \in [0, 1/2] \\ 2(1-x) & \text{if} \quad x \in [1/2, 1] \end{cases} \tag{2.4}$$

Thus, for the tent map, any point $x \in [0, 1]$ with a given itinerary $S(x)$ satisfies $\tau(S(x)) = x$.

Finally, for any unimodal map f with kneading sequence K, a symbol sequence s is allowed if and only if

$$\tau(\sigma^m(s)) \leqslant \tau(K) \qquad \text{for all} \quad m \geqslant 0 \tag{2.5}$$

This inequality is presented in a more compact form than in ref. 9, where it was first derived. An intuitive justification follows by observing that since

$c_1 = f(c_0)$ is the maximum of f, it must be true that $f^n(x) \leqslant c_1$ for all $x \in I$ and all $n \geqslant 1$. Moreover, the above construction provides an ordering on itineraries which is the same as that on the real line.

Relation (2.5) provides a simple rule to find a list or forbidden words (FW): write down the sequence $(t_1, t_2,...)$ corresponding to $\tau(K)$ and exchange the nth symbol in the kneading sequence K whenever t_n is 0. Obviously, there is at most one FW of given length n and the number of FWs of length $\leqslant n$ equals the number of 0's in $(t_1, t_2,...)$ up to the position n.[1]

For instance, the logistic map with $\mu = 4$ gives $K = 1000...$ and $\tau(K) = 111....$ In this case no FWs of any length are present, i.e., any sequence $(s_1, s_2,...)$ is admissible. Less trivial situations appear if we decrease the parameter μ. For instance, at $\mu = \mu_0 \equiv 1 + \sqrt{8.4}$ (μ_0 will be always considered throughout the paper as a paradigmatic example of typical chaos, to better describe the various approaches here presented),

$$K = 100101011001001010111101...$$

and

$$\tau(K) = 111001101110001100101001...$$

In this example the FWs of length $\leqslant 8$ are 1000,10011,10010100.

If we apply condition (2.5) to the kneading sequence K itself, we get

$$\tau(\sigma^m(K)) \leqslant \tau(K) \qquad \text{for all} \quad m \geqslant 0 \tag{2.6}$$

This means that K must be a *maximal sequence*.[6] Henceforth, let us agree to call any $\tau(K)$ satisfying (2.6) *consistent* (actually corresponding to a kneading sequence). Notice that, since the consistent τ's are related to nothing but the kneading sequences of unimodal maps, they constitute indeed a sort of universal encoding for these models. In what follows we shall be interested in characterizing the set $\Lambda \subset [0, 1]$ of all consistent τ's. Note that, by virtue of (2.3), one could even forget that each $\tau \in \Lambda$ is related to a particular kneading sequence, and construct the set Λ as the set of those numbers $\tau \in [0, 1]$ satisfying the inequality

$$T^m(\tau) \leqslant \tau \qquad \text{for all} \quad m \geqslant 0 \tag{2.7}$$

A progressive construction of this set can be obtained through a "strange repeller" approach: for any number $0 < a < 1$, representing a possible $\tau \in \Lambda$, iterate it with the tent map T and discard those a's that turn out to be nonconsistent after n iterations, according to rule (2.7). In this way, one obtains a set Λ_n which approximates Λ at order n.

The first interesting fact is the presence of "gaps" (forbidden open intervals) that appear in hierarchical order as n increases. The presence of isolated points inside the gaps is also observed. As a matter of fact, there is a one-to-one correspondence between the gaps in Λ and the periodic windows in the bifurcation diagram of unimodal maps: the lower bounds of the gaps correspond exactly to the tangent bifurcation points responsible for the birth of periodic windows, whereas the isolated points correspond to period-doubling bifurcations.

In order to understand more this point precisely, let us note that for any τ corresponding to a kneading sequence K, two different situations may occur:

(1) K is either nonperiodic or eventually periodic, and then

$$\tau(\sigma^m(K)) < \tau(K) \qquad \text{for all} \quad m \geqslant 1$$

(2) K periodic with period l, and then

$$\tau(\sigma^m(K)) < \tau(K) \qquad \text{for} \quad m = 1, 2,..., l-1$$

and

$$\tau(\sigma^l(K)) = \tau(K)$$

Clearly, the latter constitutes an extreme situation in order that (2.6) be verified. This indicates that the gaps must be somehow related to periodic kneading sequences. More precisely, consider a kneading sequence K which is periodic with period l, i.e., $K = (s_1,..., s_l, \overline{s_1,..., s_l})$ (a bar over a finite block of symbols denotes a sequence with an infinitly repeating basic block), then the corresponding $\tau(K)$ is represented by the periodic sequence $(t_1,..., t_h, \overline{t_1,..., t_h})$ whose period h is equal to l if the number of 1's in $(s_1,..., s_l)$ is even, so that $t_l = 0$, whereas $h = 2l$ if this number is odd and $t_l = 1$.

From Eq. (2.4) and from the chain rule, we have that the multiplier $\prod_{j=0}^{l-1} T'(T^j(\tau))$ is positive (negative) and >1 (<1) for any periodic orbit characterized by an even (odd) number of 1's. In the former case, this means that any point $\tau' > \tau$, in a suitably small right-neighborhood of τ, will be such that $T^l(\tau') > \tau'$, that is, nonconsistent. Therefore, any such τ defines the greatest lower bound of a gap of hierarchical level l. A similar reasoning shows that orbits with an odd number of 1's naturally represent the right band edges of the gaps of level l. However, simply by doubling the number of iterates, we see that they represent left band edges of gaps of hierarchical level $2l$ as well. This means that these τ's are *isolated* points. They are related to period-doubling bifurcation points which can be constructed without any reference to the underlying map. Let us define the pattern corresponding to a purely periodic kneading sequence as the subsequence $P_0 = (s_1, s_2,...)$ contained between two successive symbols c (for the sake of clarity, we let the critical point have its own symbol c):

$$c \to s_1 \to s_2 \to \cdots \to c$$

Of course, if $K_0 = (s_1,..., s_l, \overline{s_1,..., s_l})$ is the kneading sequence corresponding to the lower bound of a gap, then $P_0 = (s_1,..., s_{l-1})$. The pattern associated with the first harmonic K_1 is[7]

$$P_1 = P_0 \eta P_0 \tag{2.8}$$

where η is 0 if P contains an odd number of 1's, and 1 otherwise. Clearly, if τ_0 is the coding of the sequence P_0, then the coding of P_1 is given by

$$\tau_1 = \tau_0 1 \hat{\tau}_0 \tag{2.9}$$

where $\hat{\tau}$ indicates the complement (module 2) of τ. This procedure can be again applied to K_1 to obtain the second harmonics, and so forth. It is immediately realized that all the isolated points in Λ can be constructed in this way. As a consequence, as long as we consider any open interval in Λ containing at most isolated consistent points [i.e., those for which (2.7) holds] as a gap, the right gap edges are naturally identified as the period-doubling accumulation points. Practically, we proceed as follows: for each l, we find all the primitive periodic orbits with period l of the tent map T such that their symbolic sequence contains an even number of 1's. By retaining only the largest point of each orbit, we have exactly all the lower bounds of the gaps of hierarchical level l. An elementary calculation shows at once that they are rational numbers of the form $\tau = p/q$ where p is an even integer such that $2 \leqslant p \leqslant 2^l - 2$ and $q = 2^l - 1$. For instance, $l = 3$ gives one gap starting at 6/7, $l = 4$ gives again only one gap at 14/15, and $l = 5$ gives three gaps starting at 26/31, 28/31, and 30/31, respectively. One may wonder how many gaps appear at a given level n. By the way, this is relevant also as it estimates the number of periodic windows in the chaotic region of unimodal maps. A precise answer is readily obtained in the following way: the number of primitive cycles of period n of the tent map T is given by

$$P_n = \frac{1}{n} \sum_{m \mid n} \mu(m) 2^{n/m} \tag{2.10}$$

where $m \mid n$ means that m divides n, μ is the Möbius function,[10] and $2^{n/m}$ is the cardinality of the set $\{x \in I \mid T^{n/m}(x) = x\}$. Then we write $P_n =$

$P_n^+ + P_n^-$, where P_n^+ is the number of cycles of period n with positive multiplier, whereas P_n^- is the number of those characterized by a negative multiplier. Some tedious but trivial calculations show that, in this particular case, if n is odd, then P_n is even and

$$P_n^+ = P_n/2 \tag{2.11}$$

whereas if $n = 2^k q$, with $k \geq 1$ and q an odd integer, then P_q is even and

$$P_n^+ = \frac{1}{2}\left(P_n - P_{n/2} + \frac{1}{2}\left(P_{n/2} - P_{n/4} + \cdots - P_{n/2^{k-1}} + \frac{1}{2}\left(P_{n/2^{k-1}} - \frac{P_q}{2}\right)\cdots\right)\right) \tag{2.12}$$

P_n^+ is simply the number of gaps of level n (see Table I for a list up to $n = 18$). In the next sections we shall be mainly concerned with the assignment of a topological entropy to each point of Λ. It is known that this quantity remains rigorously constant over intervals which are larger than the gaps constructed with rule (2.7). In fact, as long as the asymptotic dynamics is confined inside a so-called periodic window, the topological entropy remains constant, being determined by the "external" transient

Table I. Number of Primitive Periodic Orbits of Tent Map (4.1.) for Increasing Length *n*, Compared with the Number of Gaps (i.e., Period-Doubling Sequences) and the Number of Windows (i.e., Regions Where the Topological Entropy Remains Constant)

Length	Primitives	Gaps	Windows
3	2	1	1
4	3	1	1
5	6	3	3
6	9	4	4
7	18	9	9
8	30	14	14
9	56	28	27
10	99	48	48
11	186	93	93
12	335	165	163
13	630	315	315
14	1161	576	576
15	2182	1091	1085
16	4080	2032	2031
17	7710	3855	3855
18	14532	7252	7244

structure of a repeller. The attractor structure starts being again relevant only after a collision with an unstable periodic orbit has occurred, giving rise to a sudden expansion of its size, i.e., after an interior crisis.[11]

As the most intriguing properties occur out of the windows, we restrict ourselves to this smaller set. Accordingly, we define a *window* as an interval in $[0, 1]$ over which the topological entropy h is *constant* (Fig. 2). For the sake of brevity we will use the symbol Λ to denote the set of those $\tau \in [0, 1]$ which are consistent *and* have a neighborhood where h is a non-constant function (actually increasing).

It is well known that inside each periodic window of the bifurcation diagram of unimodal maps, the whole set of bifurcations observed over the "interesting" parameter range (e.g., $\mu \in [0, 4]$ for the logistic map) is repeated exactly starting from the stable periodic orbit that defines the window. Likewise, inside each window, the whole structure of Λ repeats itself in a self-similar way. Of course, if one is interested in counting these intervals with the procedure summarized in Eqs. (2.11) and (2.12), one has to discard those windows that appear inside gaps of lower hierarchical level. This situation may occur whenever the periodic kneading sequence corresponding to the lower bound of a gap has a period l which is not prime, i.e., $l = j \cdot m$ with $j > 1$ and m prime. In this case, if $s_i = s_{l+i}$ for $i = 1,..., l-1$, then the gap is contained in a preexisting one and must therefore be discarded. In Table I the number of windows is compared with that of gaps.

The precise location of the right extrema of windows in Λ can be determined from the following considerations: let $(s_1,..., s_l, \overline{s_1,..., s_l})$ be the periodic kneading sequence corresponding to the leftmost extreme of a window. For parameter values just below the critical value corresponding to interior crises, the chaotic dynamics occurs in l distinct bands, whose boundaries are defined by the iterates of the critical point c_0 (note that c_0 is always contained in a band). When the parameter reaches the critical value, the unstable periodic orbit created at the original tangent bifurcation collides with the boundaries of the chaotic region, so that the critical point c_0 follows, after a transient, its itinerary. This means that the kneading sequence which corresponds to the very end of the window is eventually periodic and reads $(s_1,..., \hat{s}_l, \overline{s_1,..., s_l})$, where $\hat{s}_l = (s_l + 1)(\mathrm{mod}\ 2)$. In other words, if $\tau = p/q$ defines the lower bound of a window, the conjugate upper bound is

$$\tau' = \frac{2^{l-1}p + 2k - 1}{2^{l-1}q} \tag{2.13}$$

where k is an integer ranging over $(1,..., m_l)$ and m_l denotes the number of windows of hierarchical level l. Moreover, the greater is p (recall that

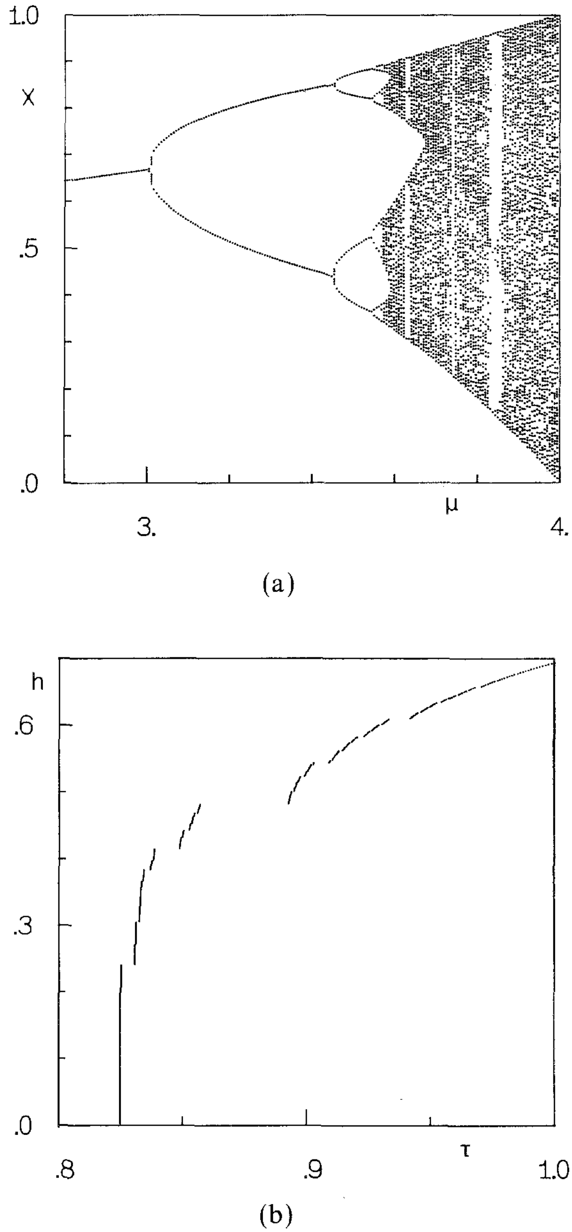


Fig. 2. (a) Bifurcation diagram of the logistic map; (b) topological entropy versus the universal parameter τ. An accurate scanning of part (a) allows one to locate the well-known periodic windows, which are immediately recognized in (b) as the regions where the entropy remains constant.

$2 \leqslant p \leqslant 2^l - 2$), the smaller is k. By the way, it is not hard to show that any rational number of the form (2.13) eventually forms a cycle of length r, where $r \leqslant (q-1)/2$, of the tent map T. By taking the difference $\tau' - \tau$ and recalling that $q = 2^l - 1$, we get

$$\Delta_l(k) = \frac{2k-1}{2^{l-1}(2^l-1)}, \qquad k = 1,..., m_l \tag{2.14}$$

namely, the width of the windows that appear at level l is not uniform: it decreases as p increases. This completes our construction of the set Λ.

3. GRAPHS AND MARKOV PARTITIONS

In the previous section we investigated the properties of the values that can be assumed by the universal parameter τ in generic unimodal maps. Here, we show how the knowledge of τ can in turn be translated into increasingly accurate descriptions of the grammatical rules of the chaotic language, by constructing suitable directed graphs.[4] The idea of using the theory of graphs to characterize dynamical systems is a very recent one and it follows from the possibility of interpreting each trajectory as a sequence of symbols. This is most obvious in the case of cellular automata, where the local variable, discrete, can be immediately considered as a symbol taken from a finite alphabet.[13] In the case of dynamical systems, instead we must first introduce a generating partition which in turn allows us to encode each trajectory as a sequence of symbols which now represents the elements of the partition visited at each time. Partial applications of graph theory to one-dimensional maps can be found in refs. 1, 3, 14, and 15.

There are various ways of constructing such machines. Here we present two alternative procedures: (i) a general one, introduced in ref. 1, which is based on the knowledge of FWs, and can be easily extended to generic symbol sequences, even not associated with dynamical systems; (ii) a more powerful one based on the knowledge of the map.

In reviewing the first procedure, we need to introduce the concept of irreducible FW.

Definition. An irreducible forbidden word is a forbidden symbol sequence which does not contain any shorter forbidden sequence.

The irreducible FWs contain the relevant information which allows one to improve the knowledge of the grammatical rules as well as of the topological entropy h. In fact, when considering sequences of increasing length, it is precisely when we encounter a new irreducible FW that we "learn" something about the grammar. Now we describe the construction of a graph, starting from the knowledge of irreducible FWs. First, let us order the FWs for increasing length, that is, once $l(F_i)$ is defined as the length of the ith FW, let $l(F_i) > l(F_j)$ for $i > j$. The equality can always be excluded, since, as already seen, there is at most one FW of length l for unimodal maps. Notice that some of the 0's in the τ expansion do not correspond to irreducible FWs, as the associated kneading sequence contains a shorter FW.

To make the construction simpler, we exploit another property of unimodal maps. The first $l(F_{j-1}) - 1$ bits of the jth FW coincide with those of the $(j-1)$th FW. This is because each FW is found by iterating the same kneading sequence. The nodes of the graph are introduced to keep track of the past symbols, in order to prevent the generation of FWs. The starting node obviously corresponds to a complete ignorance about the past history of the string. The first symbol of the first irreducible FW is associated with an outgoing arrow that points toward the second node, which in turn is identified with that symbol, whereas the arrow associated with the other symbol points to node 1 itself, as it is not leading to FWs. More in general, referring to the jth node, the addition of a generic symbol to those associated with the node itself can lead to various cases, depending upon whether the resulting sequence s_j (i) coincides with the next irreducible FW, (ii) contains a shorter FW, (iii) coincides with the first j symbols of the next FW, or (iv) other cases hold.

In cases (i) and (ii) no arrows depart from the jth node. In case (iii) the outgoing arrow points toward the new $(j+1)$th node. In the last case, a link between the jth and the mth nodes must be created, where m is determined as follows. The sequence s_j is progressively shortened, by deleting the leftmost symbol, until it coincides with the sequence associated with a previously generated node.

The procedure besides being quite general (as it can be easily extended to any sequence of symbols, without necessarily referring to 1D maps), also provides natural approximations for generic chaotic states, characterized by an infinity of irreducible FWs: it is sufficient to restrict the analysis to the FWs up to a given length n. In such cases we always find a finite graph closed onto itself. As already pointed out by Grassberger,[1] there is not immediate relation between nodes of the graph and possible elements of a Markov partition of the interval. In fact, all points corresponding to the same node share a suitable symbol sequence observed in the past, a property which cannot be unambiguously translated into belonging to a given subinterval, in noninvertible maps. Moreover, note that the existence of a finite number of irreducible FWs is a sufficient but not necessary condition to guarantee the existence of a Markov partition. The logistic map at the Misiuriewicz point (where the third iterate of the maximum falls onto the fixed point with negative multiplier) is characterized by a Markov partition,[6] although there is an infinite sequence of irreducible FWs, in particular all strings of the type $01^{2n}0$. Loosely speaking, we can say that the approximation of a generic chaotic dynamics in terms of a finite number of FWs resembles the approximation of an irrational number in terms of numbers characterized by a finite number of digits. Here, the maximum length of the strings considered plays the role of the number of digits.

The analogy with irrational numbers can be pushed even further if we follow an alternative procedure which exploits the knowledge of the underlying dynamics to determine the graph. We start by recalling that c_0 is the abscissa of the maximum, and we denote by c_i its ith iterate. The symbol 1 is associated with the interval $I_1 = (c_0, c_1)$, while 0 corresponds to a point belonging to the interval $I_2 = (c_2, c_0)$. We now associate I_1 and I_2 with the first two nodes of a directed graph. As $f(I_1) = I_1 \cup I_2$, we let two arrows depart from the first node, pointing toward nodes 1 and 2, respectively. In order to determine the evolution from the second node, we iterate I_2. If $f(I_2) = [c_3, c_1]$ cointains c_0, then $f(I_2) = I_3 \cup I_1$ (with $I_3 = [c_3, c_0]$), thus meaning that two links join node 2 with the new node 3 and node 1. In the opposite case, I_3 is defined as $f(I_2)$, and a single arrow departs from node 2 to node 3, meaning that a FW of length 2 is present. The same procedure is straightforwardly extended to I_3, and then to its iterates. From this construction it is easily seen that at least one link joining the ith node with the $(i+1)$th one is always present. Moreover, if a second arrow departing from node i exists, it points in the direction of node j, where j is the arithmetic distance from node i to the nearest node where another branching is involved. Such a property (which can be checked in the examples given in Fig. 3) will turn out to be particularly useful in the investigation of the convergence of h toward its asymptotic value. Unless an iterate c_i of the maximum falls onto a previous iterate c_j $(j < i)$, the resulting graph is infinite and we can ask which is the best way to close it onto itself to get a sequence of increasingly accurate machines. The most natural approximation is obtained by ordering the aforementioned images of c_0 from the smallest to the largest one. When considering the first i iterates of c_0, let c_{i_b} and c_{i_a} be the two iterates which are closest—from below, respectively above—to c_i (see Table II for the logistic map at $\mu = \mu_0$). It is obvious that the simplest approximations one can construct correspond to assuming $c_i = c_{i_b}$ and/or $c_i = c_{i_a}$. Such a procedure turns out to be more powerful than the previous one, as it is able to recognize all cases characterized by a finite graph (i.e., the maximum belonging to, or being eventually mapped onto a periodic sequence). This is because now we do not limit ourselves to considering the first i bits of

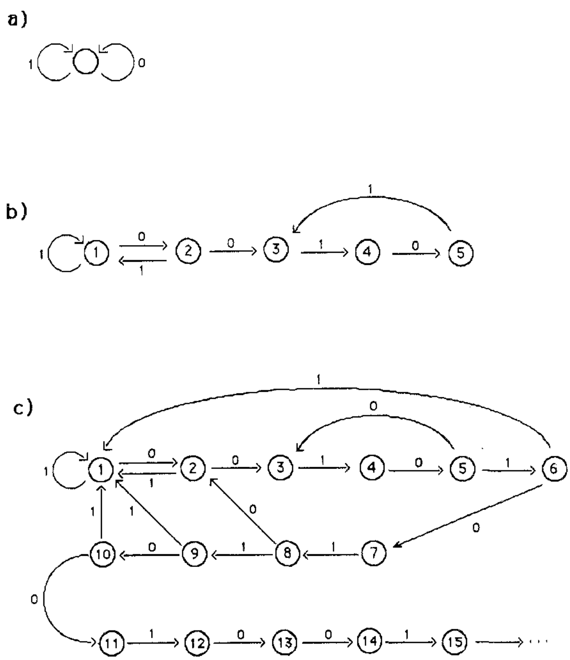


Fig. 3. Three examples of directed graphs built from the logistic map $x' = \mu x(1-x)$, for of (a) $\tau = 1$ ($\mu = 4$), (b) $\tau = 8/9$ (inside the period-three window), and (c) $\tau = 0.901903...$ ($\mu = \mu_0$).

Table II. The First 20 Iterates of the Maximum $c_4 = 1/2$ of the Logistic Map for $\mu = \mu_0$, Ordered from the Smallest (c_2) to the Largest One (c_1)

Iterate	Value	Iterate	Value
2	0.0966164	8	0.5591397
13	0.0990576	19	0.7084027
10	0.1463383	20	0.8052600
7	0.1735533	4	0.8750824
18	0.2386996	15	0.8843875
3	0.3402482	17	0.9344742
14	0.3479025	6	0.9532984
16	0.3985839	9	0.9609346
5	0.4261327	12	0.9739086
11	0.4869859	1	0.9745688
0	0.5000000		

the kneading sequence, but we exploit the knowledge of the actual value of the ith iterate of the maximum. As each approximation of the graph corresponds to choosing a suitable τ value, we list in Table III the sequence of the approximants corresponding to the two previous approaches, which confirms the superiority of this second method.

We have seen that the problem of finding an approximation of a given chaotic language corresponds to approximating the trajectory of the maximum with an eventually periodic orbit. By definition, this corresponds to finding τ values whose binary expansion is eventually periodic, i.e., rational τ's. Therefore, the above-mentioned analogy between the problem of approximating a generic chaotic state with a finite graph and approximating an irrational number with rational ones is indeed very strict. The only relevant difference is that only a tiny fraction of the numbers in [0, 1] are consistent. As a consequence, the problem of finding the optimal approximations, which is solved by the continued-fraction expansion for irrational numbers, is here much more cumbersome. In particular the problem of finding the most irrational τ is open.

So far we have discussed the problem of constructing approximate graphs to generate a grammatically correct sequence. Such a problem

Table III. Sequences of Rational Approximants of τ (Corresponding to the Logistic Map for $\mu=\mu_0$), as Determined from the Two Methods to Constuct Finite Graphs[a]

	Method 1		Method 2			
Iterates	τ	δ	τ	δ	τ	δ
3			$\frac{8}{9}$	-1.3×10^{-2}	1	9.8×10^{-2}
4	$\frac{14}{15}$	3.1×10^{-2}			$\frac{14}{15}$	3.1×10^{-2}
5	$\frac{28}{31}$	1.3×10^{-3}	$\frac{9}{10}$	-1.9×10^{-3}	$\frac{28}{31}$	1.3×10^{-3}
8	$\frac{46}{51}$	5.7×10^{-5}			$\frac{46}{51}$	5.7×10^{-5}
9			$\frac{101}{112}$	-1.2×10^{-4}		
11			$\frac{616}{683}$	-3.2×10^{-7}	$\frac{469}{520}$	1.9×10^{-5}
12					$\frac{1039}{1152}$	6.0×10^{-6}
14	$\frac{14776}{16383}$	6.8×10^{-6}			$\frac{3687}{4088}$	4.3×10^{-6}
16					$\frac{18471}{20480}$	6.1×10^{-7}
17	$\frac{118214}{131071}$	4.4×10^{-6}				
18	$\frac{236428}{262143}$	9.7×10^{-7}			$\frac{59109}{65538}$	5.5×10^{-7}
20	$\frac{945714}{1048575}$	3.0×10^{-7}	$\frac{47285}{52428}$	-3.3×10^{-7}	$\frac{354643}{393216}$	9.9×10^{-8}

[a] δ indicates the difference between the approximate and actual values of τ. Recall that the first approach yields only upper bounds to the asymptotic value.

seems to be intimately related to that of finding suitable Markov partitions, with their associated transition matrices, although in this last case, grammatical questions are not immediately involved. We have already seen that the first method to construct a graph does not indicate precisely located subintervals. The second procedure has instead shown that the nodes (or, more precisely, the links[1]) correspond to different intervals, but a problem still remains, since they are overlapping intervals, and we cannot split them without jumping into an endless zoo of intervals. However, this last method suggests a way out to the construction of an approximate Markov partition with a related graph.

Let us solve the problem into two steps. First, after having ordered the first i iterates of the maximum, and having identified c_i with the nearest c_j, we can introduce the set of intervals $I_m = [c_k, c_l]$ $(m \in \{0,..., i\})$, where c_l is the closest iterate of c_0 larger than c_k. The resulting transition matrix for the logistic map and $i = 7$ is represented in Fig. 4 as a directed graph, the only difference from previous graphs being that the links are not to be associated with the usual symbols 0 and 1. Second, we construct a graph by scanning the infinite symbol sequence backward in time. Such an operation obviously leaves the topological entropy unchanged as well as, more in general, the number of allowed strings of any length. However, we expect to find different grammatical rules for the two graphs, unless the FWs are all palindromic. Therefore, we cannot expect the graph associated with the inverted sequence to be equal to the direct one. Anyway, we can imagine inverting all FWs and applying the first procedure to build a graph. The main difference is that now the irreducible FWs do have the final rather than the starting part in common. This leads to a first more

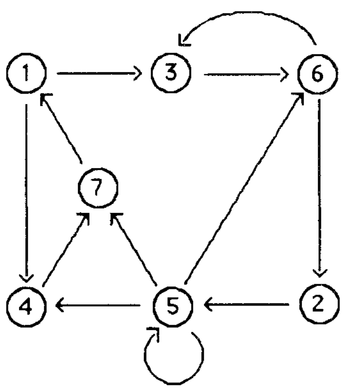


Fig. 4. Graph describing the transition matrix of an approximate Markov partition resulting from the first eight iterates of the logistic map, for $\mu = \mu_0$.

complicated graph (in Fig. 5b we present the result for our typical chaotic case, with nine nodes to be compared with the seven nodes resulting from the standard procedure, Fig. 5a). However, the Myhill–Nerode theorem[4] allows one to minimize a given graph by grouping the equivalent nodes. This is easily done by recognizing those nodes having the same "future." In the case presented in Fig. 5c, it is first recognized that nodes 6 and 9 coincide, so that nodes 4 and 8 can be identified, too, As a result, the final graph contains the same number of nodes as the one obtained by iterating forward in time (see Fig. 5c), although they remain different after any relabelling of the nodes.

Each node of the new graph corresponds to all points displaying the same suitable sequence in the future, a property which unambiguously identities disjoint subintervals of I. Accordingly, we expect a close relation with an approximate Markov partition to exist in this case. In fact, by inverting the sign of time in Fig. 5b (i.e., reversing the direction of the

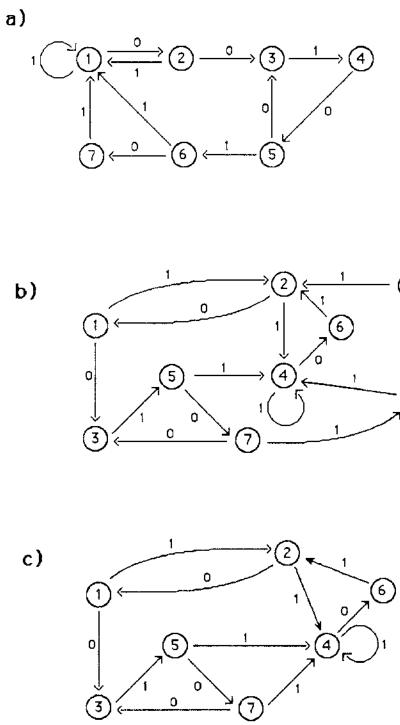


Fig. 5. (a) The graph describing the regular language characterized by the irreducible FWs 000, 0011, and 0010100, i.e., a suitable truncation of the graph in Fig. 3c; (b) graph obtained by reversing the FWs; (c) graph (b) after minimization.

arrows) we obtain a graph which is easily shown to coincide (after a suitable relabeling of the nodes) with the one in Fig. 4. The correctness of this identification is further confirmed by comparing any two equivalent nodes, i_1 in Fig. 4 and i_2 in Fig. 5b. Node i_1 corresponds, by construction, to the subinterval I_{i_1}. If we now look at the symbol sequences displayed, in the future, by all points in I_{i_1}, it is straightforwardly verified that they have in common the bits requested by the graph in Fig. 5c for all trajectories passing through the node i_2.

We can summarize the above results by recalling that two types of graph can be constructed running through an infinite sequence of symbols either backward or forward in time. The graph built by going backward is nothing but the transition matrix associated with an approximate Markov partition, upon inverting the links.

Finally, let us comment on the significance of constructing a Markov partition by going backward in time when dealing with a noninvertible transformation. Such a result is guaranteed by the existence of a unique natural extension of the dynamical system. Roughly speaking, it is an automorphism, the dynamics of which is characterized by the same graph of the original transformation.[12]

4. TOPOLOGICAL ENTROPY

Independent of the procedure adopted, a directed graph can always be interpreted as a suitable transfer matrix (the *adjacency* matrix of the graph) whose largest eigenvalue yields the topological entropy h[13,16] (see next section). However, if one is interested only in the largest eigenvalue, a superconvergent method can be introduced without resorting to the determination of any matrix. As observed in the Section 2, τ can be considered as a universal parameter. This implies that all the maps characterized by the same τ value exhibit the same topological entropy. This is also true in particular for the map

$$\begin{aligned} x_{n+1} &= m x_n & x_n &< 1/2 \\ x_{n+1} &= m(1-x_n), & x_n &> 1/2 \end{aligned} \tag{4.1}$$

which is a uniformly expanding one [one that (4.1) reduces, for $m=2$, to the tent map (2.4)]. Its Lyapunov exponent coincides with its topological entropy and it is equal to $\log m$. As a sequences, given any τ_0 deduced from the kneading sequence of a generic unimodal map, its is sufficient to determine the value of the multiplier m such that the following equality holds:

$$\tau(m) = \tau_0 \tag{4.2}$$

where $\tau(m)$ is the τ value associated with map (4.1). To give an idea of the accuracy that can be reached with little numerical effort, we give the estimated value of h for $\mu = \mu_0$,

$$h = 0.5390979671968752099740236...$$

a number accurate up to the 25th digit. The advantage of referring to map (4.1) is twofold. On one hand we have a linear map, whose topological entropy is *a priori* known; on the other hand, by changing the parameter m, only the τ values corresponding to an increasing h are scanned. More precisely, not only the admissible τ's are generated, which is obvious by construction, but the whole windows in Λ where h remains constant are automatically discarded. This is because map (4.1) is everywhere expanding so that no periodic windows in parameter space can be detected.

We are now in the position to construct the universal curve $h(\tau)$ depicted in Fig. 2. Such a curve can indeed be considered as universal insofar as we can determine τ from the kneading sequence of any unimodal map. It is sufficient to plot Eq. (4.2) considered as a functional relation between τ and $h = \log(|m|)$. The fractal-like structure of $h(\tau)$ suggests that, upon rescaling of $h(\tau)$ to $\hat{h}(\tau) = h(\tau)/h(1)$, it can be interepreted as a suitable integrated measure onto a fractal support (like, e.g., a complete devil's staircase). In other words, we argue that, whenever $\hat{h}(\tau)$ is not locally constant, it is a Hölder continuous function (of class $C^{\hat{\beta}}$ with $\hat{\beta} > 0$). This is analogous to what rigorously was proved by Guckenheimer[17] about the dependence of topological entropy on the control parameter (μ for the logistic map). As we will show, our choice of universal parameter allows us to estimate the actual value of the Hölder exponent defined by

$$\hat{\beta} = \lim_{\delta\tau \to 0} \frac{\log(\hat{h}(\tau + \delta\tau) - \hat{h}(\tau))}{\log(\delta\tau)} \tag{4.3}$$

A multifractal approach might appear as the most natural way of describing the topological entropy behavior. However, this is not the case. Indeed, although $\hat{\beta}$ depends on τ, as in a generic multifractal set, it varies in a continuous way with $\hat{h}(\tau)$, so that the most appropriate way of characterizing the set of admissible τ's is in terms of a function $\hat{\beta}(\hat{h})$.

Before proceeding in this direction, let us now give a physical interpretation of the Hölder exponent $\hat{\beta}$. It indicates the rate of change of the rescaled topological entropy when τ is changed by an infinitesimal amount. In particular, we can think of it as measuring the effect of changing the nth bit (for η sufficiently large) of the τ expansion. In other words, it evaluates the effect on h of increasing the knowledge of the kneading sequence. Thus, the existence of a finite Hölder exponent can be rephrased as the scaling hypothesis

$$h_n - h \simeq \exp(-\beta n) \tag{4.4}$$

where h_n indicates the estimate of h at the nth hierarchical level, and $\beta = \hat{\beta} h(1)$. Accordingly, β represents the exponential rate of convergence of h. Numerical simulations performed around $\tau = 6/7$ and $\tau = 52/63$ (see Fig. 6) suggest that $\beta = h$. By making use of the "equivalent" piecewise linear map (4.1), it is possible to prove that

$$\beta \leqslant h \tag{4.5}$$

Instead of fixing an infinitesimal change $\delta\tau$ of τ and looking at the entropy change δh, we look at the value of $\delta\tau$ induced by a variation δm of the multiplier m (which corresponds to a variation $\delta h = \delta m/m$ of the topological entropy). This formally corresponds to iterating the abscissa of the maximum ($x = 1/2$) with map (4.1) with two slightly different m values. By differentiating Eq. (4.1) with respect to x and m, we determine a recursive relation for the distance δx_n between the two trajectories after n iterates,

$$\delta x_{n+1} = u_n\, \delta m + b_n m\, \delta x_n \tag{4.6}$$

where b_n is the sign of the multiplier, and $u_n = x_n$ ($u_n = 1 - x_n$) if $x_n < 1/2$ ($x_n > 1/2$). A formal solution of Eq. (4.6) reads

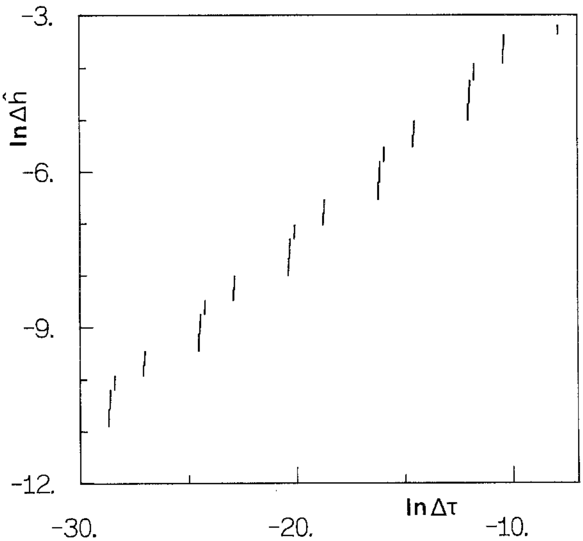
$$x_{n+1} = \sum_{i=0}^{n} \prod_{j=0}^{i} b_{n-j} u_{n-i} m^i\, \delta m \tag{4.7}$$

where the product of zero elements is defined to be equal to 1. An upper bound to Eq. (4.7) is determined by assuming all signs to the equal and setting u_{n-i} equal to the maximum value 1/2,

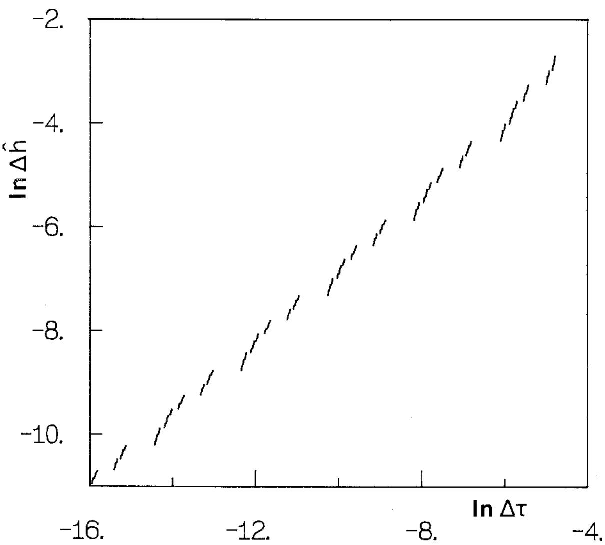
$$\delta x_{n+1} = \frac{m^{n+1} - 1}{m - 1} \frac{\delta m}{2} \tag{4.8}$$

The order of magnitude of the difference between the kneading sequences (and, in turn, of $\delta\tau$) is 2^{-n}, where n corresponds to the order of the iterate when δx_n is $\mathcal{O}(1)$. In other words, by setting the rhs of Eq. (4.8) equal to 1, then, apart from multiplicative factors, we find $\delta m \sim m^{-n}$. By finally expressing m and δm as functions of h and δh, we indeed find that h represents an upper bound to the rate of convergence β, as stated by Eq. (4.5).

Although it is quite reasonable to conjecture that β, generically, coincides with h as suggested by our numerical simulations, it is not an easy



(a)



(b)

Fig. 6. Scaling behavior of $h(\tau)$ in the vicinity of two different gap edges (a) $\tau = 6/7$ (b) $\tau = 52/63$.

matter to prove that slower rates of convergence cannot be detected. In fact, the shadowing lemma, roughly speaking, guarantees that for any trajectory generated by a hyperbolic dynamical system, it is always possible to finid a second orbit, which, iterated according to a second ("approximately equal") dynamical system, remains close to the first one. While it appears rather unlikely that both trajectories start in general from the same initial condition (as requested, since, in our problem, they have to correspond to the orbits originating from the maximum $x = 1/2$), we cannot exclude that this is the case in some exceptional circumstances. The meaning of relation (4.5) is that the topological entropy is better estimated in the most chaotic systems, where a wider set of admissible words is found. In contrast, the least chaotic dynamics are more difficult to quantify. This seemingly paradoxical result is qualitatively understood by noting that the accuracy on the estimate of h can be increased whenever the occurrence of a longer-length FW can be either discarded or taken into account. This is not the case of all the symbols that must be necessarily inserted in the kneading sequence for consistency reasons. Since in the least chaotic systems (where τ is smaller) these cases are more frequent, we have to go through a larger number of bits before reaching a preassigned accuracy.

The question on the convergence rate of h_n is analogous to, though different from, the convergence of block entropies, considered by György and Szepfalusy[18] and by Grassberger.[19] The first, marginal, difference is that they were interested in metric rather than topological entropies. Therefore, let us first give the definition of the topological block entropy,

$$H(n) \equiv \log N_{n+1} - \log N_n \tag{4.9}$$

where N_n is the number of admissible sequences of length n. As is well known, h is the limit

$$h = \lim_{n \to \infty} H(n) \tag{4.10}$$

The main difference between $H(n)$ and h_n is immediately seen by referring to a chaotic state characterized by a finite number of irreducible FWs. In such a case, the correct value of h is exactly recovered from the corresponding finite graph, while $H(n)$ converges only asymptotically to the correct estimate. This is because the second eigenvalue $\exp(h^{(2)})$ will be in general nonzero, indicating

$$N_n \simeq a_1 \exp(hn) + a_2 \exp(h^{(2)}n) \tag{4.11}$$

(here, for the sake of simplicity, we assume the second eigenvalue to be real, too) so that $h - h^{(2)}$ measures the rate of convergence of the block entropies. Notwithstanding this more relevant difference between h_n and $H(n)$, we have to register the analogy between the convergence rate β introduced in this paper (equal to the topological entropy) and the convergence rate of metric block entropies which was shown to be equal to half of the metric entropy in ref. 18. Accordingly, this difficulty of providing a quantitative characterization of weakly chaotic systems confirms the idea proposed by some authors[13,19,20] that the most complex behavior has to be expected between the two extrema of ordered and fully random states.

5. SPECTRAL PROPERTIES

In this section we concentrate on the spectral properties of the directed graphs constructed above. As far as *topological* features are concerned, all the relevant information about dynamics is stored in the adjacency matrix A whose entries a_{ij} are given by

$$a_{ij} = \begin{cases} 1 & \text{if nodes } i \text{ and } j \text{ are connected by an arrow } i \to j \\ 0 & \text{otherwise} \end{cases} \tag{5.1}$$

It follows directly from the definition and from a rapid glance at the structure of these graphs that $a_{ii} = 1$ only if $i = 1$, $a_{ij} = 0$ if $j > i + 1$, and $a_{ij} = 1$ if $j = i + 1$. For $j < i$, $a_{ij} \neq 0$ whenever a cycle of period $i - j + 1$, in which i and j are consecutive nodes, occurs. To summarize, the generic structure of the adjacency matrix for graph Γ with n nodes is

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ a_{21} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & 0 & 1 & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & 0 \end{pmatrix} \tag{5.2}$$

Note that the number of entries $\neq 0$ in the lower triangular submatrix (including the diagonal) is nothing but the number of cycles of period $1 \leqslant p \leqslant n$ occurring in Γ. Since the structure sketched in (5.2) corresponds to a particular labeling of the nodes in the graph, it is clear that we shall be interested primarily in those properties of A which are invariant under permutations of the rows and columns. Foremost among such properties are the spectral properties. For instance, the topological entropy of the mapping associated with A is given by

$$h = \log \lambda_{\max} \tag{5.3}$$

where $\lambda_{\max}$ is the largest eigenvalue of A. This can be intuitively understood by observing that, generically, in the regular binary language generated by a graph, only some number N_l of the 2^l possible sequence of l symbols may occur. This number is provided exactly by the sum of all the elements of $\mathbf{u}_l = A^l \mathbf{u}_0$, where $\mathbf{u}_0$ is the vector specifying the intial condition, with all the elements equal to 0 except for the first one, corresponding to the initial node, equal to 1. As all the elements of $\mathbf{u}_l$ are positive, their sum constitutes a norm, so that, for large l,

$$N_l = \|\mathbf{u}_l\| \sim \lambda_{\max}^l \tag{5.4}$$

and (5.3) follows from

$$h = \lim_{l \to \infty} \frac{1}{l} N_l \tag{5.5}$$

The whole set $\{\lambda_i\}$ of eigenvalues of A is determined from the characteristic polynomial $\chi(\lambda)$. This is an interesting invariant that can be written in the form

$$\chi(\lambda) = \lambda^n + c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \cdots + c_n \tag{5.6}$$

Note that the coefficient of λ^n is 1, so that the largest (real) root of $\chi(\lambda)$ is always an algebraic integer. Furthermore, it is well known that for each $k \in \{1,..., n\}$ the number $(-1)^k c_k$ is the sum of those principal minors of A which have k rows and k columns. This means that (1) since the diagonal elements of A are all zero except for a_{11}, $-c_1 = 1$, and (2) a principal minor with two rows and two columns, and which has a nonzero entry, must be of the form

$$\begin{vmatrix} a_{ii} & 1 \\ 1 & 0 \end{vmatrix}$$

where $a_{ii} = 1$ if $i = 1$ and $a_{ii} = 0$ otherwise. There is one such minor for each irreducible 2-cycle occurring in Γ, and each has value -1. Hence $(-1)^2 c_2 = -$(number of 2-cycles in Γ).

Iterating this procedure, one easily finds the following interpretation of the coefficients c_k:

$$c_k = \sum (-1)^j N_k^j \tag{5.7}$$

where the sum is over all the combinations of distinct nonrepeated cycles of Γ whose total length sums up k, and N_k^j is the number of such combinations with j elements.

This formula turns out to be very useful in that it can provide a further, and more general, justification to the scaling relation for the topological entropy [see Eqs. (4.4) and (4.5)]. Indeed, suppose we have constructed a graph Γ_n which takes into account all FWs up to length n. Then, suppose we discover the existence of another (nontrivial) FW of length $>n$. The most significant change in the estimate of h is found when the length of the new FW is $n+1$. Therefore, we restrict ourselves to this simplest case. Accordingly, from the discussion presented in Section 3, the $(n+1)$th node of the graph Γ_{n+1} returns back to the k'th node, with $k' \leqslant k+1$, where k labels the node where the nth node of Γ_n returns back. In other words, if

$$\chi(\Gamma_n; \lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_n$$

is the characteristic polynomial of Γ_n, then the first $n-k$ coefficients of $\chi(\Gamma_{n+1}; \lambda)$ remain unchanged and

$$\chi(\Gamma_{n+1}; \lambda) = \lambda\chi(\Gamma_n; \lambda) + \Delta_k \tag{5.8}$$

where Δ_k is a polynomial of degree at most k. In particular, if $k' = k+1$, Δ_k is simply the constant c_{n+1}. Let us denote by λ_* the solution of $\chi(\Gamma_n; \lambda_*) = 0$, and by $\lambda_* + \delta$ the solution of $\chi(\Gamma_{n+1}; \lambda_* + \delta) = 0$ (obviously, $\delta < 0$). By assuming δ small, we can write $\chi(\Gamma_n; \lambda_* + \delta) \sim \delta\chi'(\Gamma_n; \lambda_*)$, with $\chi'(\Gamma_n; \lambda_*)$ of order λ_*^{n-1}. Putting everything together and solving for δ, we obtain

$$|\delta| \sim \Delta_k \lambda_*^{-n} \tag{5.9}$$

If k, for $n \to \infty$, remains finite, which is true for typical chaotic cases where the trajectory frequently returns to the initial nodes, Δ_k provides an irrelevant correction to the prefactor, and we recover the result $\beta = h$. Again, we are faced with the problem that nongeneric parameter values can exist where a slower convergence is expected.

Furthermore, formula (5.7) is useful in that it allows us to identify a set of invariants entirely determining the spectrum of A. We point out that the knowledge of the whole set $\{\lambda_i\}$ of eigenvalues of A would be equally informative, as it summarizes in a compact fashion the essential information about the number N_l of admissible finite blocks for all l's. As we have seen above, the way in which this information is summarized passes through a suitable storage of some *fundamental periodic configurations* occurring in the graph. For instance, the characteristic polynomial of the graph reported in Fig. 3a is simply

$$\lambda - 2 = 0$$

giving the well-known result $h = \log 2$. A more interesting example is provided by the graph shown in Fig. 3b, originating from a τ located inside the period-3 window. In this case one finds

$$\lambda^5 - \lambda^4 - \lambda^3 - \lambda^2 + \lambda + 1 = 0$$

and this polynomial can be decomposed as

$$(\lambda^2 - \lambda - 1)(\lambda^3 - 1) = 0$$

Note that the left-hand part is the characteristic polynomial of the first transient subgraph in Fig. 3b (up to noded 2). It gives two eigenvalues at $\lambda = (1 \pm \sqrt{5})/2$. On the other hand, the right-hand part is the characteristic polynomial of the recurring subgraph in Fig. 3b (involving nodes 2, 4, and 5), which gives a threefold degenerate eigenvalue at $\lambda = 1$. The largest eigenvalue is provided by the transient subgraph, and the topological entropy is $h = \log((1 + \sqrt{5})/2)$. This is indeed a typical situation: whenever a graph is constructed from a τ located inside a window, it can be decomposed into a transient part, which provides the largest eigenvalue, and a final one. The transient subgraph is the same throughout any given window: it is defined once and for all by the periodic kneading sequence corresponding to original tangent bifurcation. So, the largest eigenvalue of the whole graph does not care at all how complicated the final structure of the graph is. This frozen situation ends when the largest eigenvalue of the final subgraph collides with the largest eigenvalue of the transient part, i.e., when an interior crises occurs. This explains the constancy of the topological entropy inside the periodic windows.

Our last example is devoted to a comparision between the spectral properties of the graphs shown in Figs. 4, 5a, and 5c. Although these graphs are quite different from each other, it easy to check, by virtue of Eq. (5.7), that they exhibit the same characteristic polynomial,

$$\lambda^7 - \lambda^6 - \lambda^5 - \lambda^4 + \lambda^3 + \lambda^2 - \lambda - 1 = 0$$

whose eigenvalues are

$$\lambda_{max} = 1.71565$$

$$\lambda_{1,2} = 0.801255 \pm i \cdot 0.515914$$

$$\lambda_{3,4} = -0.475503 \pm i \cdot 0.986406$$

$$\lambda_{5,6} = -0.683578 \pm i \cdot 0.260686$$

This suggests that the different methods of constructing graphs discussed in Section 3 are effectively consistent: they yield the same spectral properties.

Finally, let us note that just as the number of finite blocks N_l for all l's may be summarized in the characteristic polynomial $\chi(\lambda)$, so also the number $N(p)$ of cycles of period p may be stored in a topological ζ-function[21]

$$\zeta(z) = \exp\left[\sum_{p=1}^{\infty} \frac{z^p}{p} N(p)\right] \tag{5.10}$$

Since $N(p)$ is the trace of A^p, one has at once that

$$\zeta^{-1}(z) = \det(1 - zA) \tag{5.11}$$

so that the inverse of the roots of $\chi(\lambda)$ are the poles of $\zeta(z)$. In particular, the infinite series in (5.10) converges for $|z| < e^{-h}$. Furthermore, from (5.11) we have that whenever the graph is finite (i.e., for regular languages), ζ is a rational function,[22] being essentially $1/\chi$. Again, among the infinite set of the $N(p)$, it is a certain number of fundamental configurations that outline the analytic structure of ζ.

Let us mention that in a more general context (where one may associate a weight to each cycle, so as to investigate metric properties as well) some authors have recently constructed a nice procedure to evaluate ζ-functions analogous to (5.10). Briefly, this procedure consists in expressing ζ^{-1} as a *cycle expansion* where terms are grouped into dominant contributions and curvature corrections due to nonlinearities.[23] In this approach, whenever some pruning rules are present (i.e., the set of FWs is not empty), cycle expansions are written by redefining the alphabet (so as to implement such rules) and considering any possible sequence in the new language as allowed.

As a matter of fact, in those situations where FWs of considerable length are present, this seems to be a very hard task, sometimes quite hopeless. In our opinion, at least insofar as topological properties are concerned, a way out is provided by the procedure here described: whatever the original dynamical systems may be, once the set of FWs is known, the graph constructed from such a set provides directly those fundamental configurations that turn out to control the spectral features of the dynamics. If the set of FWs is infinite, the accuracy of any finite approximation is ruled by (5.9).

6. CONCLUSIONS

Throughout the paper we have seen how all relevant information about the topological features of a "generic" chaotic state can be implemented with increasing accuracy in terms of suitable directed graphs.

For unimodal maps this is a particularly simple task. However, for more general dynamical systems, the construction of an accurate encoding procedure is still a question far from being exhaustively answered. Of course, the knowledge of the topological graphs here discussed can be exploited even further. For instance, we claim that the distribution of the zeros of the characteristic polynomial χ (see Section 5), if suitably interpreted, would be informative in that it would provide a global characterization of the topological complexity of the dynamics. We shall report elsewhere on detailed studies of the spectral properties of graphs associated with unimodal 1D maps as well as with some 2D maps for which a method to extract the set of FWs is now available.[5]

Furthermore, even though an appropriate encoding technique is given, so that topological graphs can be produced, it still remains to set up an efficient procedure to extract also metric information from such graphs. In principle, this should be done by associating a probabilistic weight to each node of the graph according to the region in phase space (e.g., the element of some partition) to which it can be suitably related. The correspondence between graphs and Markov partitions (or approximate versions of them) discussed in Section 3 provides some reassuring items to this purpose.

Such an extension of the procedure described in this paper would give, for instance, an alternative scheme for constructing finite-dimensional approximations of the Perron–Frobenius–Ruelle transfer operator, whose eigenvalues allow one to determine a variety of physically interesting averages.

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